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A THEOREM for finding the Surface of an OBLIQUE CYLINDER, with its Geometrical Demonstration. Also, an Appendix, containing some Observations on the Methods of finding the Circumference of a very Eccentric Ellipse; including a Geometrical Demonstration of the remarkable Property of Elliptic Arcs discovered by Count Fagnani. By the REV. J. BRINKLEY, A. M. M. R. I. A. Andrews Professor of Astronomy in the University of Dublin.

Read Dec. 20, 1802.

DR. Barrow in his “Geometricæ Lectiones”* remarks the difficulty of finding the surfaces of an oblique cylinder, and of an oblique cone, the base of which are circles. Mentioning the former, he speaks of, “insuperabilis illa difficultas, quâcum conflictantur, qui cylindricas obliquas superficies conantur dimetiri, seu cum cylindricis superficiebus rectis aliisve quoadtenus cognitis superficiebus quoad proportionem comparare.”—Having obtained a very neat and simple theorem for the surface of an oblique cylinder, which, if I am not mistaken, is new, I am induced to submit it to the Academy, the more particularly, as it admits a demonstration purely geometrical.

* Lect. 2.

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“ The surface of an oblique Cylinder is equal to a rectangle contained by the diameter of its base and the circumference of an ellipse, the axes of which are the length and perpendicular height of the cylinder.”

This Theorem is more readily investigated from fluxional principles than from considerations purely geometrical; yet I believe no author has hitherto communicated it even so derived. The geometrical demonstration is according to the method of the ancients by means of circumscribed and inscribed prisms.

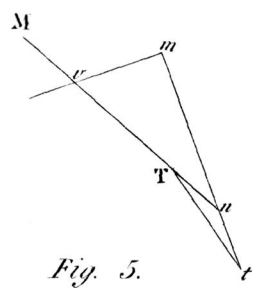
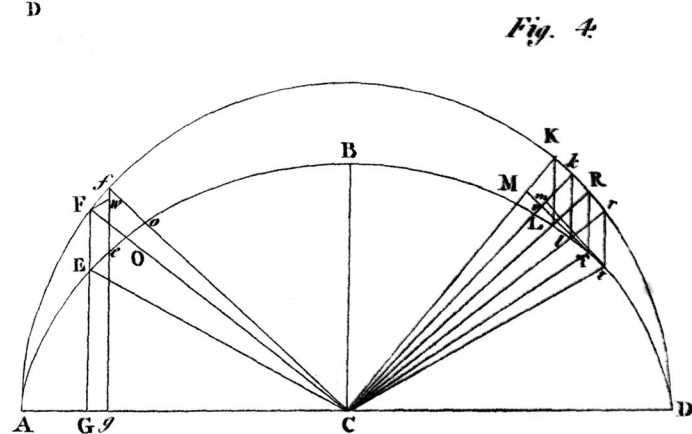
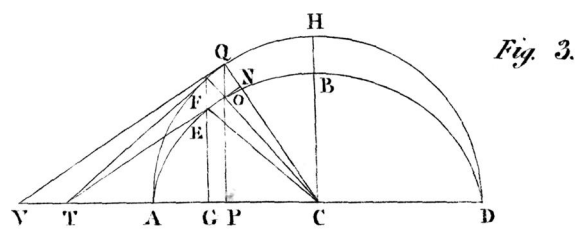
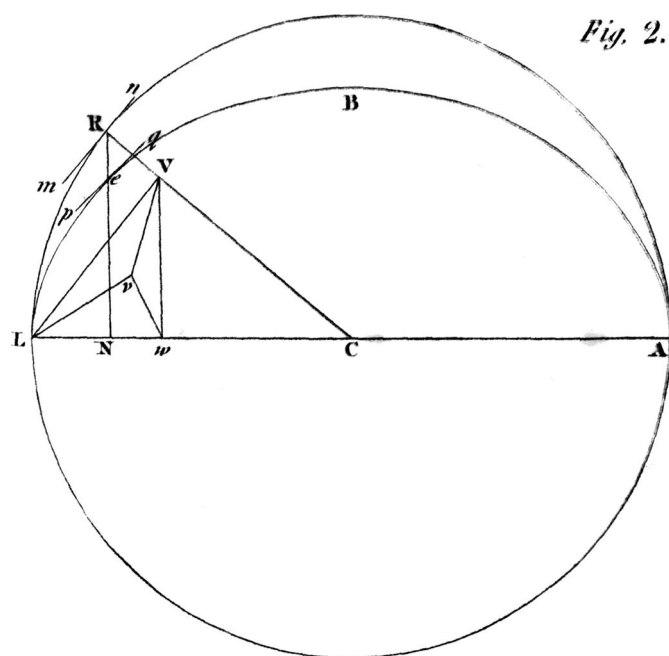
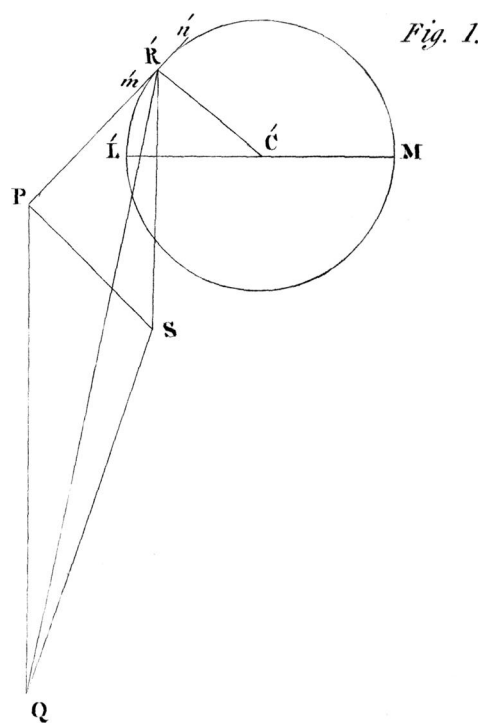
As a necessary supplement for applying this theorem in practice, some observations on the methods of obtaining the circumference of a very excentric ellipse are given. The circumference of an ellipse approaching to a circle is readily found, but that of an ellipse very excentric requires much greater research. The best method perhaps of obtaining the circumference of such an ellipse is by the assistance of the theorem of Count Fagnani. This remarkable theorem has been investigated heretofore by the application of algebra and fluxions. But by help of a curious, and I believe, new property of the ellipse, it admits of a simple geometrical demonstration; and I am enabled to derive the following theorem.

If that semi-diameter of an ellipse be taken, which is a mean proportional between the semi-axes, and be produced to meet the circumscribing circle; then the point, where the ordinate to the circle drawn from the point of intersection cuts the ellipse, divides the quadrantal arc of the ellipse into two parts, the difference of which is equal to the difference of the semi-axes.

THEOREM.

The surface of an oblique cylinder is equal to a rectangle contained by the diameter of its base and the circumference of an ellipse, the axes of which are the slant side and height of the cylinder.

DEMON-



DEMONSTRATION.

Let the circle $\acute{L} \acute{R} M \acute{L}$ represent the base of the cylinder, $Q \acute{R}$ its flant side, and QS its perpend. height. Draw the diameter $\acute{L} M$ perpendicular to $\acute{R} S$ also SP perpendicular to the tang. $\acute{R} P$, and then QP will be perpendicular to PR . Let the tangent $\acute{m} \acute{R} \acute{n}$ be the side of a polygon which is the base of a prism circumscribing the cylinder. Fig. 1

Let LRA be a circle, the diameter of which is equal to $Q \acute{R}$ the flant side. Conceive this circle inclined to the plane of its orthographical projection LBA in an angle equal $\acute{R} QS$ the complement of the inclination of the cylinder. Take the angle $LCR = \acute{L} \acute{C} \acute{R}$. Let mn be the side of a polygon circumscribing the circle, similar and similarly situate to the polygon of which $\acute{m} \acute{n}$ is a side, and let pq be the projection of mn . Draw LV parallel to the side of the polygon mn , also draw Lv the projection of LV , Vw and RN perpendicular to CL , and join vw . Fig. 2

Then the right angled triangles $Vv w$ and $\acute{R} QS$ are similar, because Fig. 1 and 2
 $RQS = Vwv$ (by constr.) : and also the right angled triangles $P \acute{R} S$ and RNC are equi-angular.

$$\text{Hence } Vv : Vw :: \acute{RS} : \acute{RQ} \\ Vw : RN = LV :: CV = CN : RC :: PR : R\acute{S}$$

$$\text{Therefore } Vv : LV :: PR : \acute{RQ}$$

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whence

whence, the angles $V \text{ } v \text{ } L$ and $Q \text{ } P \text{ } R$ being right angles, it follows that the triangles $L \text{ } v \text{ } V$ and $P \text{ } Q \text{ } R$ are similar,

and that $PQ : QR = AL :: Lv : LV :: pq : mn$

but $AL : LM :: mn : m'n'$ by similar figures.

therefore $PQ : LM :: pq : m'n'$ or rect. $PQ \cdot mn = LM \cdot pq$. or the side of the prism = rect. $pq \cdot LM$ and consequently the whole surface of the circumscribing prism = rectangle $LM \cdot \times$ the circumference of the polygon circumscribing the ellipse, which polygon is the projection of the polygon circumscribing the circle.

In the same manner it may be proved that the surface of a prism inscribed in the oblique cylinder is equal to the rectangle contained by LM and the circumference of the polygon inscribed in the ellipse.

And as this is true of any whatsoever circumscribed and inscribed prisms, it follows that the surface of the cylinder is also equal to the rectangle contained by the diameter of the base LM and the circumference of the ellipse which is the projection of the circle, for otherwise, as may be easily shewn, an absurdity might be deduced Q.E.D.

APPENDIX.

The quadrature of the surface of an oblique cylinder is reduced, as above, to the rectification of an ellipse. The circumference of an ellipse is, as is well known, equal to the circumference of the circumscribing circle

$$\times : 1 - \frac{e^2}{2 \cdot 2} - \frac{3e^4}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5e^6}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} * \&c. \text{ } e \text{ being the excentricity to}$$

the

* A series converging much more rapidly when e is small, is given in Mr. Ivory's very ingenious Essay, Edinburgh Transactions, Vol. 4. 1798.

the semi-axis major 1. But when the excentricity is considerable, this series, converging slowly, ceases to be useful. Several geometers of the first rank have particularly considered this difficulty. The result of their researches being connected with the complete solution of the above problem, properly requires some notice here.

If x be the distance of an ordinate from the centre of the ellipse, the fluxion of the elliptic arc intercepted between the ordinate and extremity of the axis minor is $\frac{x}{\sqrt{1-x^2}} \times \sqrt{1-e^2 x^2}$. This fluxion is easily re-

duced to the form $\frac{Px}{\sqrt{a+bx+cx^2+dx^3+ex^4}}$ P being a rational function of

x . The eminent mathematicians Euler, Lagrange, and Legendre, have employed themselves on this form. Lagrange has been particularly successful, and by a most ingenious process has shewn that it may in every case be transformed so that its fluent may be obtained by swiftly converging series. His memoir on this subject is to be found in the "Mem. Acad. des Sci. en. de Turin 1784, 1785," and his method is justly styled by Lacroix * "La plus elegante peut etre qui soit sortie de la plume des analystes." However, in the case of the excentric ellipse this method does not furnish so simple a solution as may be otherwise derived. For in the application of Lagrange's method, a remarkable theorem offers itself, † by which the circumference of one ellipse may be derived from that of two others less excentric. The first discovery of this theorem is due to Legendre, and was given, derived by a different method, in the 2d of his two very ingenious essays on elliptic arcs. ‡ He had first discovered that the circumference of one ellipse may be derived from the circumference of another ellipse by means of partial differences, and afterwards combining this conclusion with a method

* Traite du calcul. diff. et integral, vol. 2. p. 88.

† Lacroix. vol. 2. Art. 506, 507

‡ Mem. Acad. 1786.

method pointed out by Mr. Landen's discovery that an hyperbolic arc may be rectified by means of two elliptic arcs, he derived the above-mentioned conclusion. This theorem enables us to derive the circumference of a very excentric ellipse from the circumferences of two ellipses, the excentricities of which may be as small as we please.

Let b, b', b'' &c. be the semi-axes minors of a series of ellipses, the semi-axes majors of which are unity; so that $b' = \frac{4b}{1+b^2}$, $b'' = \frac{4b'}{1+b'^2}$, &c. Then

the rectification of two adjacent ellipses of this series being known, the rest are easily had by the above-mentioned theorem. Let E, E', E'' represent quadrants of three adjacent ellipses of this series, the respective excentricities of which are e, e', e'' then
$$\frac{2E' - (1+e')E}{1-e'^2} = \frac{1+e''}{2} \cdot \frac{4E'' - 2(1+e'')E'}{1-e''^2}$$

The terms of the series b, b', b'' rapidly approach to unity, so that the rectification of a very excentric ellipse, is reduced to the rectification of two of small excentricity to be performed by the common theorem.

But when the ellipse is very excentric, a series may be obtained of as easy application as the common series, and therefore is to be preferred to the above methods.

This series is given by Legendre, and is derived by him from an application of the remarkable and elegant theorem discovered by Count Fagnani. The methods Legendre used to obtain this series and its law are strikingly ingenious, and probably will not admit of improvement. For the method and law of the series I refer to the memoir.*

The

* Mem. Acad. 1786.

I learn from a very ingenious memoir of Mr. Wallace (Edinb. Transf. vol. 5 p. 267) in which the general rectification of the ellipse is particularly treated of, that Mr. Euler gave the same series in a work which I have not seen, entitled, "Animadversiones in rectificationem ellipsis." Mr. Wallace also, in his memoir, has given an elegant formula for the rectification of an ellipse.

The application of Fagnani's theorem arises from the following circumstance. While the series for the whole quadrantal arc of an excentric ellipse, is useless from its slow degree of convergency, a part of the quadrantal arc commencing at the extremity of the axis minor, may be found by a series sufficiently converging, and the remainder of the elliptic quadrant (the series for which would be diverging) may be obtained from its relation to the first arc, by that theorem.*

Therefore, as connected with the above subject, I have subjoined a geometrical demonstration of Fagnani's theorem; which demonstration is derived from a property of the ellipse. which I believe to be new.

LEMMA.

Let AED be a semi-ellipse on the axis major AD, C the centre and AFD the circumscribing circle. Let also any ordinate GE be produced, to meet the circle in F. Draw FC intersecting the ellipse in O and QOP parallel to FG, then if QC be drawn intersecting the tangent EN in N, CN will be perpendicular to the tangent EN, and equal to CO.

Demonstration.

Draw the tangents FT and TE which intersect in a point T of CA produced. The triangles POC and FTG are similar, therefore
TG

* It may be proper to remark an error in a passage of the "Excerpta ex epistolis Newtoni" noticed, I believe, by none of the commentators. Two series are given for computing the length of an elliptic arc, and in finding the length of the quadrant, the semiaxis is directed to be bisected, and the arcs corresponding to the two abscissas to be found by the two series (Page 312, art. 7, vol. 1. Horsley's edit.) But the direction in this passage is, as may be readily shewn, entirely impracticable, whenever the ratio of the axis major to the axis minor exceeds the subduplicate ratio of 5 : 3. For then one of the series will be diverging. When the excentricity is considerable, no division of the semiaxis major will render the application of the Newtonian series useful, as the convergency of one or other will not be greater than that of the common theorem for the whole circumference of an ellipse.

$$TG : GF :: PO : PC$$

and $GF : GE :: PQ : PO$ by the ellipse.

whence $TG : GE :: PQ : PC$ and consequently the triangles TGE and PQC are equiangular, and the angle $PCQ = TEG$; therefore the angles at G and N are together equal two right angles, and so the angle at N a right angle.

Now draw QV parallel to TN, and

$$CN : CQ :: CT : CV :: (\text{because } CV \times CP = CQ^2 = CF^2 = CG \times CT) \\ CP : CG :: CO : CF \text{ and therefore as } CQ = CF, CN = CO. \text{ Q.E.D.}$$

Cor. When CO is a mean proportional between AC and CB, CE = the femi-conjugate to CO, and also $EN = AC - CB$.

Demonstration.

$CN \times CO = CO^2$ (by hypothesis) $AC \times CB$. $\therefore$ CO = the femiconjugate to CE: and therefore $AC^2 + CB^2 - 2AC \times CB = CE^2 - CN^2 = EN^2$ and therefore $EN = AC - CB = Q. E. D.$

THEOREM.

Fig. 4. AEOBD is an ellipse, the axis major of which is AD. Produce any ordinate GE to meet the circumference of the circumscribing circle in F. Draw COF and CT a semiconjugate to CO, and also the tangent TM meeting the perpendicular CM in M: then arc BT = arc AE = TM.

Demonstration.

Draw fg indefinitely near to FG. Let Ct be semiconjugate to Co and draw TR, tr parallel to the axis minor BC. Let also CR, Cr meet the ellipse in L, l . Draw LK, lk parallel to the axis minor, join K, M; k, C and draw the tang. tm meeting k, C , which intersects TM in

in v . Then by the lemma it easily appears that K, M, C are in one right line, and also that m is a right angle.

Because FCK and fCk are each right angles, therefore $Ff = Kk$. Also if Fw be drawn parallel to the tangent at E , the triangle Ffw is ultimately similar to the triangle CLK ; for CL is parallel to the tangent at E , because FCK is a right angle, and therefore CE and CL are conjugate to each other; and also CK is parallel to a tang. Fig. 4. at F .

Therefore ultimo $Ff = Kk :: Fw :: CK : CL = CM$ by lemma.

Also ultimo $Mv : Kk :: CM : CK$

consequently ultimo $Mv = Fw$. But ultimo

$Mv^* = TM - tm + \text{arc } Tt$ and $Fw = Ee$, therefore

ultimo $Tt - Ee = tm - MT$ or ultimo the increment of BT — increment of $AE = \text{increment of tang. } MT$. Therefore as these magnitudes begin together the arc $BT - \text{arc } AE = TM$. Q. E. D.

Cor. If CO be taken a mean proportional between AC and CB , and be produced to meet the circle in F , then drawing the ordinate FG , the point of intersection E will divide the elliptic quadrant, so that $BE - AE = AC - BC$.

Demonstration.

By this construction $CE =$ the semiconjugate to CO (cor. to lemma) and therefore $CE = CT$, and therefore $BE = BT$. Whence $TM = \text{tang. at } E =$ (by cor. lemma) $AC - BC$. Consequently by the theorem, arc $BE - \text{arc } AE = AC - BC$. Q. E. D.

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* Let the tang. TM, tm intersect in n . Then $Mv = MT - vT = MT + Tn - vn$. Fig. 5. But ultimo $vn = mn = tm - tn$ therefore ultimo $Mv = TM - tm + Tn + nt =$ ultimo $TM - tm + \text{arc } Tt$.

It may appear strange after all that has been written upon the method of prime and ultimate ratios (or perhaps more properly speaking of limits) that with respect to a demonstration in which that method is used, more should be deemed necessary than merely a reference to it, as already established upon logical principles. However of late, eminent mathematicians both at home and abroad have again called in question the principles of the doctrine, and rather than use a method so admirably adapted to the purposes intended by the illustrious inventor, compendium of invention and demonstration, have had recourse to methods which fall little short of the tedious demonstrations of the ancients. From hence it should seem that its principles have not been put by its defenders, and commentators in that clear light in which they ought to be placed; at least, judging from the eminence of its present opponents. Without acceding to such an opinion, there can, I think, be no impropriety in endeavouring to shew that a demonstration in which that method is used, is unimpaired thereby. Principally with this intention the following summary of the doctrine is given, in which is more particularly noted that part of it introduced into the above demonstration. The whole is contained in effect in the first section of the Principia. But the great author, pressing forward to more important matters, did not stop to give that precision to the doctrine which has been since required by its opponents. He was satisfied with anticipating some objections, and giving some cautions, which, if fully attended to, will enable any one to place the method on the most solid foundations.

It being granted that the doctrine is founded upon logical principles, no objection can be made to its use in demonstrations either purely geometrical or analytical.

The superiority it possesses over other methods whether ancient or modern, with respect to facility of invention and demonstration, appears most satisfactorily by comparing the processes instituted according to the respective methods.

This superiority I consider as a sufficient apology for introducing the controversy respecting its principles. It is not convenient here to enquire particularly how the opponents of the doctrine have understood those points, which they conceive are unanswerable objections. If I do not deceive myself, the method as deduced from what the illustrious author has left us, is capable of the most logical proof. What is here given is by no means intended as a complete summary. If it serves to shew that the above application of the method is logical, and also in any degree to shew, that the general principles of the method are neither obscure nor inaccurate, the purpose is answered. Not that I can flatter myself with the hope of advancing much new illustration, but as the old objections have again been brought forward in another shape, I think it incumbent upon those, who imagine a method so important to be logically founded, to endeavour, if they cannot advance new defences, to bring forward the old ones, in a manner likely to be most effectual.

The method of limits may be considered as reducible to the following heads.

I. The

1. The definition of a limit. 2. Problems, by which limits so defined are obtained. 3. Properties of those limits. 4. Application of those properties to geometrical demonstrations, to algebraical processes, and to the translation of physical relations into mathematical ones.

1. *Definition* A limit of a variable quantity, whether a simple magnitude or the magnitude of a ratio is that quantity to which the variable quantity may so approach as to differ from it in magnitude by less than any assigned or stated quantity.

1. A variable secant drawn from a given point without a circle, may be drawn, so as to differ from the tangent in magnitude, less than by any magnitude that can be assigned. For, assign a difference, then the position of the secant is determined, and therefore between the secant and tangent another secant can be drawn. The magnitude of the tangent is a *limit* of the magnitude of the secant. But the secant is not said to become the tangent.

2. A limit of the quantity $\frac{x^2 - a^2}{x - a}$ where x is greater than a is $2a$ for $\frac{x^2 - a^2}{x - a}$ may be made to differ from $2a$ by a less magnitude than any assigned one. But $2a$ cannot with propriety be said to be one of the values of $\frac{x^2 - a^2}{x - a}$ or to be equal to $\frac{0}{0}$.

3. The limiting ratio of the increment of the abscissa of a curve to the increment of the ordinate is equal to the ratio of the subtangent to the ordinate. The ratio of the increments themselves is never equal to the ratio of the subtangent to the ordinate.

II. *Prob.* It is therefore a mathematical problem to find a limit so defined, of a variable quantity, when it admits of one. The solution of the problem is evidently had if the variable quantity or its equal can be expressed by the sum or difference of two quantities one fixed and the other variable, the variable one admitting a less value than any assigned one. The fixed quantity is a limit.

1. Supposing x any magnitude greater than a , and a limit of $\frac{x^2 - a^2}{x - a}$ be required.

Because $\frac{x^2 - a^2}{x - a} = x + a = 2a + e$ (putting $x = a + e$). Therefore as e may be less than any assigned magnitude $2a$ is the limit required.

To determine the limiting ratio of increment of the abscissa to the increment of the ordinate. The incr. of the abscissa : increm. ordinate :: subtangent $+ v$: ordinate, by similar triangles. As the ordinates may approach so that v may become less than any assigned quantity, the limit of the latter, and therefore of the former ratio is the ratio of the subtangent to the ordinate. This method is frequently applicable when the terms of the proposed ratio are both variable. A ratio is found equal to the given ratio, one of the terms of which is fixed and the other variable. A limit of the variable term is then found, and thence the limit of the proposed ratio is had.

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In this manner Sir Isaac Newton investigated the limiting ratio of the arc to the chord, tang. &c. In this way also the limiting ratio of the increments of the algebraical quantities x and x^n is had. Let o be the increment of x , and $nx^{n-1}o + n \cdot \frac{n-1}{2} x^{n-2}o^2 + \&c.$ is the increment of x^n ; the general ratio then of these increments is equal to the ratio $1 : nx^{n-1} + n \cdot \frac{n-1}{2} x^{n-2}o + \&c.$ In this series it may readily be shewn that o may be taken so small that the sum of the terms after the first may be less than any quantity that can be assigned; so that, according to the definition $1 : nx^{n-1}$ is the limiting ratio required.

Here is no shifting of the question, as it has been termed, by first taking o a real magnitude and then no magnitude, nor any introduction of infinitesimal quantities. The question is concerning the limit of the ratio of the increments not concerning the ratio itself; the general ratio of the increments is equivalent to a ratio, one term of which is fixed, and the other composed of a fixed, and variable quantity; this variable quantity being susceptible of a less value than any that can be assigned, the ratio of the fixed quantities is by the definition the limiting ratio.

2. For the ready solution of the problem, the following proposition is often of the greatest importance, and is one of the principal sources of compendium derived from this doctrine.

Prop. A. In deducing a limit of a variable quantity, depending upon other variable quantities, the limits of these variable quantities may be used instead of the quantities themselves in any part of the process.

Let L be the limit of any variable quantity M involved in the process; then in any of the steps L may be used for M , and the conclusion will be true. For let $\epsilon = M - L$ express the general relation between L and M , then, without considering the limit, in the conclusion will be found instead of L , $L + \epsilon$.

But to obtain the limit required, the limits of the variable quantities must be used, and therefore now L substituted for $L + \epsilon$.

This proposition relates both to geometrical magnitudes and analytical quantities.

Any steps, which we can demonstrate will lead us to a true conclusion, must be considered with a reference to that conclusion, as logical. Such are the steps in which we substitute limits for the quantities themselves, although the quantities are never accurately equal to their limits. By subjoining the word *ultimò* to such steps we refer to the general conclusion, and intend these steps are only true with a reference to that. From being thus enabled to shorten the steps of a demonstration arises much of the value of the method, whether it be applied to geometrical demonstration or analytical processes.

This

This proposition is assumed in the third cor. of the 7th, and in the cor. of the 8th lemma, 1st Book of the Principia.

III. From the properties of limits, the three following may be selected, as frequently used; of which the second, as being more connected with the above theorem, is particularly considered. The others easily follow from the definition of a limit.

1. Quantities which are the same limits of the same variable quantity, are equal to one another.

2. Prop. B. If the limiting ratio of the corresponding increments of two magnitudes commencing together, be always a ratio of equality; the magnitudes themselves are in a ratio of equality.

For: Let M and N be the two magnitudes, and let M be divided into *any* number n of equal parts p , and let also N be divided into the same number of corresponding parts q, r, s, t , &c. Then the parts of N are all equal to each other, or some of them are unequal.

If they are always all equal $M : N : p : q$, and the limiting ratio of the increments must be the same as the constant ratio of the increments themselves, and therefore $M : N$ is the ratio of equality.

If the parts of N are not equal to each other; one of them must be greatest and one of them must be least; therefore, taking *any* number, if q be greatest, and r least, nq is greater than N , and nr less. If $M : N$ be not the ratio of equality, let it differ from the ratio of equality by the ratio $e : f$. Now the ratio of $M : N$ is between the ratios $np : nq$, or $p : q$ and $np : nr$ or $p : r$, consequently one of them must differ from the ratio of equality by a ratio greater than the ratio $e : f$; but by hypothesis the ratios $p : q, p : r$, &c. may be nearer the ratio of equality than by any assigned ratio. Hence the ratio $M : N$ cannot differ from a ratio of equality. (Vid. cor. 4 Lemma Prin. Math.)

3. If an equation exists between the corresponding variable quantities $L + e, L' + e',$ &c. where e, e' &c. may be less than any assigned magnitude, however small, L, L' &c. being fixed, and therefore the limits of these quantities; then, if in that equation L, L' &c. be substituted for $L + e, L' + e',$ &c. an equation will be had between the limits.

IV. The following instances will serve for illustrating the application of these principles.

1. In the demonstration of the above geometrical theorem, the *limiting* ratio of the increment of the difference of the elliptic arcs to the increment of the tangent is deduced by the application of the principles of prop. A. From the limiting ratio of these increments, the ratio of the difference of the arcs to the tangent is deduced by prop. B; not from the ratio of the evanescent increments, but from the limiting ratio of the increments.

2. The following method of finding the sine in terms of the arc, shews the applications in an algebraic process.

Let s = the sine of arc a to rad. 1, and let $\sin \frac{a}{n} = \frac{a}{n} - e$, n representing any assigned number, however great. Then by a well known theorem for the sign of a multiple arc

$s = n \cdot \frac{a}{n} - e - \frac{n \cdot n^2 - 1}{2 \cdot 3} \cdot \frac{a}{n} - e^3 \&c. = a - ne - 1 - \frac{1}{n^2} \cdot \frac{a - ne}{2 \cdot 3} + \&c.$ Now the limit of $a - ne$ is a for the limiting ratio of $\frac{a}{n} : \frac{a}{n} - e$ (limiting ratio of arc to sine) is the ratio of equality, therefore the limit of the equal ratio $a : a - ne$ is the ratio of equality, that is as a is fixed, the limit of $a - ne$ is a . Hence it readily appears, taking the limits of the terms of the above series, $s = a - \frac{a^3}{2 \cdot 3} \&c.$

This equation is not deduced by neglecting quantities as infinitely small; but because the first equation is true when n is any assigned number however great, the last must necessarily be true also for otherwise it is easily shewn, that the first could not be generally true, when n is any assigned number however great.

3. The ratio of the centripetal force in two points of a curve is the limiting ratio of the sagittæ. The limiting ratio of the sagittæ in the ellipse, the force tending to the focus, is the inverse duplicate ratio of the distances. And as the ratios which are the limits of the same variable ratio must be equal to each other, the ratio of the forces is equal to the inverse duplicate ratio of the distance.

The application of limits to physical enquiries strikingly illustrate the value of the method. The beautiful instances in the Principia Mathematica of Newton are too well known to dwell upon. The slightest comparison will in those instances, shew the superiority of this method over those that have lately been brought forward.